Collins' notes on Lemmon's Logic

(i) Rule of Assumption (A)
Insert any formula at any stage into a proof. The assumed formula rests upon the assumption of itself.

(ii) Double Negation (DN)

a. $\dfrac{\neg\neg A}{A}$ b. $\dfrac{A}{\neg\neg A}$

('Two negations cancel each other out'.)
The derived formula rests upon the assumptions upon which the transformed formula rests.

(ii) &-Introduction (&I)

$$\frac{A \quad B}{A \,\&\, B}$$

('One can derive the conjunction of any two preceding formulae')
The derived formula rests upon the assumptions upon which the individual conjoined formulae rest.

(iii) &-Elimination (&E)

$$\frac{A \,\&\, B}{A \qquad B}$$

('One can derive either conjunct form a preceding conjunction')
Either of the derived formulae rest upon the assumptions upon which the conjunction rests.

(iv) v-Introduction (vI)

$$\frac{A}{A \vee B} \qquad \frac{A}{B \vee A}$$

('One can derive a disjunction which has as a disjunct any preceding formula')
The derived formula rests upon the assumptions upon which the given disjunct rests.

(v) v-Elimination (vE)

$$\begin{array}{c} \underline{A \vee B} \\ \underline{A} \quad \underline{B} \\ \underline{C \quad C} \\ C \end{array}$$

('One can derive a formula from a disjunction so long as one can derive the formula from both disjuncts of the initial disjunction')
The derived formula rests upon the assumptions of (i) the initial disjunction, (ii) what each conclusion from each disjunct assumption rests upon *minus* the assumptions of each disjunct..

(vii) Modus Ponendo Ponens (MPP)

$$\frac{\begin{array}{l} A \\ A \rightarrow B \end{array}}{B}$$

('One can derive the consequent of a conditional if the antecedent of the conditional is also given')
The derived formula rests upon the assumptions upon which the conditional and its antecedent rest.

(viii) Conditional Proof (CP)

$$\begin{array}{l} \underline{A} \\ \underline{B} \\ A \rightarrow B \end{array}$$

('One can derive a conditional from the assumption of its antecedent if one can derive the conditional's consequent from the assumption')
The derived formula rests upon the assumptions upon which the assumption and its derived formula rest, *minus* any assumptions upon which both rest. (I.e. the assumption of the antecedent drops out.)

(ix) Reductio ad Absurdum (RAA)

$$\begin{array}{l} \underline{A\quad\quad} \\ \underline{\neg A \,\&\, A} \\ \neg A \end{array}$$

('One can derive the negation of an assumption if, from that assumption, one can derive a contradiction')
The derived formula rests upon the assumptions upon which the initial assumption and the contradiction rest, minus any assumptions upon which both rest. (I.e. the initial assumption drops out.)

(x) Equivalence definition (↔df)

a. $\dfrac{A \leftrightarrow B}{A \rightarrow B \,\&\, B \rightarrow A}$ b. $\dfrac{A \rightarrow B \,\&\, B \rightarrow A}{A \leftrightarrow B}$

First-order Logic adds

(i) *Universal Elimination* (UE)
For any formula, either derived or assumed, '$(\forall x)Fx$', which contains at least one occurrence of 'x' bound, one can derive 'Fa', which rests on all the assumptions upon which '$(\forall x)Fx$' rests.
Justification: If everything is F, then any arbitrary object (a) will also be F.

(ii) *Universal Introduction* (UI)
If one can derive 'Fa' resting upon no assumptions which feature 'a', then one can derive '$(\forall x)Fx$', which binds every occurrence of 'x' substituting 'a', and rests upon all assumptions upon which 'Fa' rests.
Justification: If F holds of an arbitrary object (a) resting upon no assumptions about the object, then F holds of everything.

(iii) Existential Introduction (EI)
For any formula, either derived or assumed, ‘Fa’, one can derive ‘(∃*x*)F*x*’, which binds every occurrence of ‘x’ substituting ‘a’, and rests upon all assumptions upon which ‘(∃*x*)F*x*’ rests.
Justification: If an arbitrary object (a) is F, then something or other is F.

(iv) Existential Elimination (EE)
For any formula, ‘(∃*x*)F*x*’, if, from assumption ‘Fa’, where ‘a’ substitutes every occurrence of *x*, one can derive C, then one can conclude C, resting upon all assumptions upon which ‘(∃*x*)F*x*’, ‘Fa’, and C rest, minus any assumptions upon which two or more formulae rest (in effect, this will mean that C follows from ‘(∃*x*)F*x*’ minus ‘Fa’. [Ed's comment: that is what Collins gives, but I think Lemmon had a restriction here that 'a' should be a new name, not one that has already been used in the derivation.]
Justification: if C follows from an arbitrary object (a) being F, then it follows from something or other being F.

(i) *Identity introduction* (=I)
At any stage of a proof, one introduce ‘a = a’ as a logical truth which rests upon no assumptions.
Justification: Everything is identical with itself as a matter of logic, so ‘a = a’ need rest on no assumptions.

(ii) *Identity elimination* (=E)
If, as assumptions or derived formulae, ‘Fa’ and ‘a = b’, then one may derive ‘Fb’, resting upon all assumptions upon which ‘Fa’ and ‘a = b’ rest.
Justification: Since identities share all properties, if a = b, then b has any properties a does.

Strategic issues

Finding a natural deduction proof is a matter of discovery rather than the mechanical following of a rule. One technique that is often very helpful is to work backwards. Given what the conclusion is, how can one get it, using the rules? Given what one needs to get the conclusion, how can one get that, etc. etc.?

If the conclusion is a conditional, then one expects to conclude with a conditional proof, so start by assuming the antecedent.

If the conclusion (or what one needs at some stage) is a negation then reductio ad absurdum may well be the way to go.

Derived rules

Natural deduction systems become more user-friendly since they allow you to use derived rules: once you have shown that a particular form of argument is valid, you can use that form as a derived rule.

Target forms

This terminology I have borrowed from a forgotten source; it is simply a generalisation of the idea of derived rules to cover not only valid forms of argument but invalid ones. Traditionally people learned certain simple forms of argument, some valid, some invalid, which they used as standards to match particular arguments against. Once the form fits a particular argument you can say that (assuming the fit is adequate) the argument is valid or invalid.

Besides Lemmon's basic rules, these are some useful target forms, with their traditional names:

VALID

Modus tollens:

$\varphi \rightarrow \psi$
$\neg \psi$
so $\neg \varphi$

Disjunctive syllogism:

$\varphi \vee \psi$
$\neg \varphi$
so ψ

Hypothetical syllogism:

$\varphi \rightarrow \psi$
$\psi \rightarrow \chi$
so $\varphi \rightarrow \chi$

Quantifier shift

$(\exists x)(\forall y)Rxy$
so $(\forall y)(\exists x)Rxy$

Equivalences -- one can argue in either direction:

Contraposition:

$\varphi \rightarrow \psi \Leftrightarrow \neg \psi \rightarrow \neg \varphi$

De Morgan's laws

$\varphi \vee \psi \Leftrightarrow \neg (\neg \varphi \wedge \neg \psi)$ equivalently $\neg (\varphi \vee \psi) \Leftrightarrow \neg \varphi \wedge \neg \psi)$

$\varphi \wedge \psi \Leftrightarrow \neg (\neg \varphi \vee \neg \psi)$ equivalently $\neg (\varphi \wedge \psi) \Leftrightarrow \neg \varphi \vee \neg \psi$

Importation (left to right)/Exportation (right to left)

$\varphi \rightarrow (\psi \rightarrow \chi) \Leftrightarrow (\varphi \wedge \psi) \rightarrow \chi)$

Distributive laws:

$\varphi \vee (\psi \wedge \chi) \Leftrightarrow (\varphi \vee \psi) \wedge (\varphi \vee \chi)$

$\varphi \wedge (\psi \vee \chi) \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$

Material conditional equivalences:

$$\varphi \rightarrow \psi \Leftrightarrow \neg \varphi \lor \psi \Leftrightarrow \neg (\varphi \land \neg \psi)$$

INVALID

Denying the antecedent:

$\varphi \rightarrow \psi$
$\neg \varphi$
so $\neg \psi$

Affirming the consequent:

$\varphi \rightarrow \psi$
ψ
so φ

no name (related to denying the antecedent)

$$\varphi \rightarrow \psi \Leftrightarrow \neg \varphi \rightarrow \neg \psi$$

Quantifier shift

$(\forall y)(\exists x)Rxy$
so $(\exists x)(\forall y)Rxy$

There are many other examples of valid and invalid operator shifts
Valid:
It is known that not P, so it is not known that P

Either it is necessary that P or it is necessary that Q, so it is necessary that either P or Q

It is possible that (both P and Q), so it is possible that P and it is possible that Q

There is someone who is obliged to F, so it is obligatory for someone to F

Invalid:
It is not known that P, so it is known that not P

It is necessary that either P or Q, so either it is necessary that P or it is necessary that Q [replace 'Q' here with 'not P' and you have a fallacious argument for fatalism]

It is possible that P and it is possible that Q, so it is possible that (both P and Q)

It is obligatory for someone to F, so there is someone who is obliged to F