Symbolic Logic 5

Designation

We have called items that pick out objects or individuals names. A lot of problems arise because in English and other languages many items corresponding to our names are used to do other things than pick out individual objects. The logic of such uses is complicated and we shall not try to capture it. We might, however, note (with Hodges) some of the types of use we will *not* be dealing with.

Hodges offers a rough test for when a name is purely referential: rewrite the sentence containing it (let us say, D) in the form:

D is a person (thing) who (which, such that) … he (she, it) …

If that sentence says the same as our original then D is purely referential.

Designators or names can fail to be purely referential in various contexts:

- Temporal contexts, e.g. 'it's unprecedented for *the Managing Director of Kiki Products* to be Japanese'. '*My husband* used to play in the band' - if used by a re-married woman to refer to her former husband.

- Modal contexts (possibilities or necessities), e.g. If *the Managing Director* had been chosen by ballot, he'd be some flashy whizz-kid.'

- Intentional contexts (beliefs, hopes, ..), e.g. Smith never realised that Mr Hashimoto was *the Managing Director.*'

- Quotational contexts, e.g. 'Mr Hashimoto likes to be referred to as *the Managing Director.*'

Meaning of quantifiers in ordinary English

Hodges suggests that we can see the semantic content of quantifier phrases as falling into one of four types of profile.

Suppose we have two predicates (S and P), then a quantifier may tell us:

1. How many Ss are P (and say nothing about how many Ss are not P), e.g., no Ss are P; many Ss are P, two Ss are P, ...

2. How many Ss are not P, but nothing about how many are P, e.g., all Ss are P, every one of the Ss is P, …

3. What proportion of the Ss are P, e.g., half the Ss are P, nearly all Ss are P, …

4. Definite descriptions tell us that just one S is P, e.g., John is the tallest boy in the class.

To formalise some definite descriptions we need an extra item in our language: a sign for identity. With the power identity gives us we can translate quantifiers of profiles 1, 2, and 4 fairly well into our language; we can’t deal with profile 3.

Identity

Identity (usually written ‘=’) is a special relation in our language. It says that D_1 is the very same things as D_2. It can be used to translate claims like ‘Everest is the highest mountain in the world’, ‘Cassius Clay and Muhammed Ali are the same person’, etc.

The rules for its use in trees are two:

1. The sentence $\neg$ (a = a) is inconsistent (so it closes any branch it is on). (We can write such sentences as ‘a $\neq$ a’.)

2. The other rule permits substitutions based on identity claims, e.g. given that a = b, and Fa we can move to Fb, and vice versa.

Expressive power due to identity

If our language includes identity we can translate quantifiers that speak of particular numbers of things, at least *n*, at most *n*, exactly *n*. The existential quantifier is understood to say that there is at least one. Given that and identity, we can specify other numbers.

To illustrate:

> To say ‘there are at least two Gs’, we say ‘$\exists x\ Gx \wedge \exists y\ Gy \wedge x \neq y$’.
>
> To say ‘there are at most two Gs’, we can say ‘there are not at least three Gs’ ‘$\neg(\exists x\ Gx \wedge \exists y\ Gy \wedge \exists z\ Gz \wedge x \neq y \wedge x \neq z \wedge y \neq z)$’.
>
> To say ‘there are exactly two Gs’ we conjoin the previous two claims, or slightly shorter we can write: ‘$\exists x\ Gx \wedge \exists y\ Gy \wedge x \neq y \wedge (\forall z)(Gz \rightarrow (x = z) \vee (y = z))$’.

We can also say ‘there is exactly one G’ in a briefer way: ‘$\exists x\ \forall y\ (x = y \leftrightarrow Gy)$’.

Obviously, given how complex these formulae become, we use abbreviations if we are actually dealing with precise numerical quantifiers. But the point is that we can say exactly what ‘there are 101 Dalmations’ says, just using our mini-language with identity.

As noted above, another increase in expressive power afforded by identity is the ability to capture what is important about many uses of definite descriptions. What we understand by a sentence like ‘The S is P’ is that there is precisely one S under discussion and every S under discussion is P, so we are going to translate it as in effect ‘there is exactly one S, and every s is P’:

$$\exists x\ \forall y\ (x = y \leftrightarrow Sy) \wedge (\forall x)(Sx \rightarrow Px)$$

We have to be careful about the domain of quantification here, but usually that isn’t a serious problem.

Similar paraphrases allow us to deal with superlatives, for instance. ‘Harrow United is the best team’ becomes ‘Harrow United is a team and for all x, if x is a team and x $\neq$ Harrow United then

Harrow United is better than x'.

Relations, properties of

Ordered n-tuples of objects satisfy relations. An n-place relation is a set of ordered n-tuples on a domain. Can specify a relation by listing its ordered n-tuples, or by providing a condition things have to satisfy in the domain.

Graphs of binary relations: draw a circle for the domain, and dots for each object; for each ordered pair <b, c> draw an arrow from b's dot to c's. Make it a double arrow if also < c, b>. Make it a loop if < b, b>.

A binary relation is

> *reflexive* if every dot has a loop - $\forall x\ Rxx$
> *irreflexive* is no dot has a loop - $\forall x\ \neg Rxx$
> *non-reflexive* if neither reflexive nor irreflexive.
>
> *symmetric* if no arrow is single - $\forall x \forall y\ (Rxy \rightarrow Ryx)$
> *asymmetric* if no arrow is double - $\forall x \forall y\ (Rxy \rightarrow \neg Ryx)$
> *non-symmetric* if neither symmetric nor asymmetric

Call a route from b to d via c a broken journey if there is an arrow from b to c and one from c to d. If there is an arrow from b to d there is a short cut.

> *transitive* if no broken journeys without a short cut - $\forall x \forall y \forall z\ (Rxy \wedge Ryz) \rightarrow\ Rxz)$
> *intransitive* if no broken journeys with a short cut - $\forall x \forall y \forall z\ (Rxy \wedge Ryz) \rightarrow\ \neg Rxz)$
> *non-transitive* if neither transitive nor intransitive.
>
> *connected* if any two dots are connected by an arrow - $\forall x \forall y\ (\ x \neq y \rightarrow (Rxy \vee Ryx))$

A relation that reflexive, symmetric and transitive is an *equivalence* relation. Can construct abstractions from such equivalence relations: hardness; capacity (volume); personality; …