Logical Truth and Axiomatic Approaches to Logic

Logical Truth

We have met the idea that to every argument there corresponds a conditional statement (if the premises then the conclusion) and we have seen that in the case of truth-functional logic this conditional will be a tautology when the argument is deductively valid. A tautology is true in every possible case, true no matter how things happen to be: a truth of logic rather than fact.

When looking at trees we also allowed for the case where there are no premises in an argument. If we could show that the negation of a statement led to a closed tree then the unnegated statement would be true no matter what, again a truth of logic. But it is easier to see such statements arising within a natural deduction system, since from any given case where a conclusion can be derived from a set of premises, a sequence of conditional proofs (if premise$_1$ then conclusion, if premise$_2$ then if premise$_1$ then conclusion, …) eventually yields a conclusion resting on no assumptions whatever, again true no matter what.

While every valid argument yields logical truths in this way there are other logical truths that do not correspond to actual arguments. Among the most common are some principles that have been acknowledged from the earliest times: the law of non-contradiction ($\neg (\varphi \wedge \neg \varphi)$) and the law of excluded middle ($\varphi \vee \neg \varphi$).

Axiomatic theories

Ever since Euclid, the axiomatic method has regularly been used to present a body of theoretical claims about a subject-matter, especially in mathematics. Though Euclid did not specify them in the way we now would, an axiomatic system contains rules of inference, definitions, and a set of axioms. Axioms are claims about the subject-matter, from which, by the rules of inference and the definitions, the other truths about the subject-matter can be derived.

Irvine (in SEP) has written "During the critical movement initiated in the 1820s, mathematicians such as Bernard Bolzano, Niels Abel, Louis Cauchy and Karl Weierstrass succeeded in eliminating much of the vagueness and many of the contradictions present in the mathematical theories of their day. By the late 1800s, William Hamilton had also introduced ordered couples of reals as the first step in supplying a logical basis for the complex numbers. In much the same spirit, Karl Weierstrass, Richard Dedekind and Georg Cantor had also all developed methods for founding the irrationals in terms of the rationals. Using work by H.G. Grassmann and Richard Dedekind, Guiseppe Peano had then gone on to develop a theory of the rationals based on his now famous axioms for the natural numbers. Thus, by Frege's day, it was generally recognized that a large portion of mathematics could be derived from a relatively small set of primitive notions." The question that exercised Frege was whether these mathematical axioms could themselves be derived from purely logical claims. It was thus that he constructed a logical theory as an axiomatic theory of the logical truths, rather than as an account of processes of valid inference.

From Frege's work (1879) to the middle of the 20th century the predominant approach to the study of logic was via axiomatic systems, typically with a small number of axioms or axiom-schemas and a very small number of rules of inference. As we noted with natural deduction systems with several rules of inference, finding a proof using just the axioms and the allowable rules of inference can be very difficult.

One issue for axiomatic systems is that the axioms be independent (so none are superfluous). Another relates to coverage - can we derive all the relevant truths from the axioms? And another to whether it is decidable whether a given statement is logically true (or derivable from the rules).

With respect to these issues, truth-functional logic is different from first-order logic. Truth-functional logic is complete and decidable. In general first-order logic is not decidable. But there are weaker systems of first-order logic, e.g. the monadic predicate calculus,[1] that are decidable.

It is possible to prove that first-order logic is complete in the sense that if a formula is logically true then there is a finite deduction (a formal proof) of the formula.

Another obvious issue for an axiomatic theory is that it be consistent, i.e. you can't derive a contradiction from the axioms. Given their soundness (that they only yield logical truths) the completeness results for various logics imply consistency. But in general to prove the consistency of one theory you need a stronger theory. Gõdel proved in particular that any logic strong enough to do regular arithmetic cannot prove itself to be consistent (if it is consistent – if it is inconsistent it can 'prove' anything).

1 In this logic there are only monadic predicates (other than identity). It corresponds roughly to the types of statement that Aristotle formalised in his syllogistic.